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The Weighted Residual Method in Solution of More General Nonlinear Integro-Fractional Differential Equation of Volterra Hammerstein Type with Variable Coefficients

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Abstract

In this paper, we solve numerically Volterra-Hammerstein (V-H) Integro-Differential Equations (IDE's) of various fractional order $n-1<\alpha\le n$ ($n\in\mathbb{Z}^+$) in the Caputo sense by the well-known (Collocation, Sub-domain, Least-square, and Galerkin) weighted residual method. This method is utilized to converts the integro-fractional differential equation to a matrix equation which corresponds to a system of nonlinear algebraic equations with unknown polynomial coefficients. A good algorithm for treating numerically our problem by applying the above process has been developed, in order to express these solutions, programs are written in MatLab (V.8). The validity and reliability of this method are tested by several illustrative numerical examples with the known exact solution, the behaviors of the error are examined, and the obtained results reveal that the method is more accurate and efficient.

Introduction

The goal of this paper is to provide a numerical scheme for solving a multi-high fractional order in Caputo sense of nonlinear Integro-Differential Equations (IFDE) of Volterra-Hammerstein (V-H) type with variable coefficients in the general form on the interval $[a,\mathcal{T}]$:

$$ {}_a^C D_t^{\alpha_n} u(t)+\sum_{i=1}^{n-1} p_i(t)\, {}_a^C D_t^{\alpha_i} u(t)+p_0(t)u(t)=f(t)+\lambda\sum_{\ell=0}^{m}\int_a^t \mathcal{K}_\ell(t,s)\,\mathcal{H}_\ell\left(s\,,{}_a^C D_s^{\beta_\ell}u(s)\right)ds \qquad (1)$$

Under the following initial conditions

$$u^{(k)}(a)=u_k\ ;\ k=0\,,1\,,2,\cdots,\mu-1 \qquad (2)$$

Where $\mu=max\{n_i\,,m_\ell$ for all i and $\ell\}$ and $u_k\in\mathbb{R}$ for all k. Where $u(t)$ is the unknown function which is the solution of equation (1) under initial conditions (2), as well as, the functions $\mathcal{K}_\ell: S\to\mathbb{R}$ with $(S=\{(t,s): a\le s\le t\le\mathcal{T}\}\ ;\ \mathcal{H}_\ell: S_*\times\mathbb{R}\to\mathbb{R}\ (S_*=\{s: a\le s\le t\ ;\ t\le\mathcal{T}\});\ \ell=0\,,1\,,\cdots,m$, and $f,p_i:[a,\mathcal{T}]\to\mathbb{R}\ ;\ i=0,1,2,\cdots,n$ are all continuous functions. In addition, $\alpha_i\,,\beta_\ell\in\mathbb{R}^+\,,n_{i-1}<\alpha_i\le n_i$; $m_{\ell-1}<\beta_\ell\le m_\ell\,,n_i=\lceil\alpha_i\rceil$; $m_\ell=\lceil\beta_\ell\rceil$ for all $i=\overline{1:n}$ and $\ell=\overline{0:m}$ with property that: $\alpha_n>\alpha_{n-1}>$

$\cdots > \alpha_1 > \alpha_0 = 0$ and $\beta_m > \beta_{m-1} > \cdots > \beta_1 > \beta_0 = 0$ and λ is a scalar parameter. ${}_a^C D_t^\gamma$ Denotes the Caputo fractional derivative of order $\gamma \in \mathbb{R}^+$.

The nonlinear Integro-Fractional Differential Equation is an important branch of modern mathematics. Such equations occur in various areas of applied mathematics, physical phenomena and bioengineering [12,13,18]. Volterra-Hammerstein is one of important types, which arises in various branches of applications such as Free Electron Laser. More recently, applications have included classes of nonlinear equations [6,13] and this motivates us to consider their effective numerical solution.

Expansion method using the weighted residual (error) technique, including Collocation, Sub-domain, Least Square and Galerkin methods are one of the most popular minimizing techniques that used to determine the expansion parameters. Many authors and researches have consider these methods to solve differential, fractional differential, integral and integro-differential equations [1,3,4,5,7,11,14,19,20]. In this paper, we extend this technique to further deal with consider problem (1) with given initial conditions (2).

The paper is organized as follows. Section 2 presents the necessary definitions and basic preliminaries of the fractional calculus; section 3 devoted to formulation of weighted residual method for solving nonlinear IFDE of V-H type; our results illustrated throughout examples in section 4. Finally, section 5 includes a discussion for this method.

Fractional Calculus:

Fractional calculus deals with the differentiation and integration of arbitrary order, many definitions of it have been proposed. Most frequently occurring are Riemann-Liouville and Caputo. In this section, we present the definition and important properties of these two types of fractional derivatives, which are used throughout this paper. We begin by defining the function space $C_\gamma, \gamma \in \mathbb{R}$, which was used in development of the operational calculus for the differential operator, [13,16,18].

Definition 1: [17]

A real valued function u defined on $[a,b]$ be in the space $C_\gamma[a,b]$, $\gamma \in \mathbb{R}$, if there exists a real number $p > \gamma$, such that $u(t) = (t-a)^p u_*(t)$, where $u_* \in C[a,b]$, and it is said to be in the space $C_\gamma^n[a,b]$ if and only if $u^{(n)} \in C_\gamma[a,b], n \in \mathbb{N}_0$.

Definition 2: [13]

Let $u \in C_\gamma[a,b], \gamma \geq -1$ and $\alpha \in \mathbb{R}^+$. Then the Riemann-Liouville fractional integral operator ${}_aJ_t^\alpha$of order α of a function u, is defined as:

$${}_aJ_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\xi)^{\alpha-1} u(\xi) d\xi \ \ , \alpha > 0$$

Moreover, if $\alpha = 0$, then ${}_aJ_t^0 u(t) = Iu(t) = u(t)$.

Definition 3: [18]

Let $\alpha \geq 0$, and $m = \lceil \alpha \rceil$. the Riemann-Liouville fractional derivative operator ${}_a^R D_t^\alpha$, of order α and $u \in C_{-1}^m[a,b]$ and defined as:

$${}_a^R D_t^\alpha u(t) = D_t^m \, {}_aJ_t^{m-\alpha} u(t)$$

If $\alpha = m$, $m \in \mathbb{N}_0$, and $u \in C^m[a,b]$ we have, ${}_a^R D_t^0 u(t) = u(t)$; ${}_a^R D_t^m u(t) = u^{(m)}(t)$.

Definition 4: [21]

The Caputo fractional derivative operator ${}_a^C D_t^\alpha$ of order $\alpha \in \mathbb{R}^+$ of a function $u \in C_{-1}^m[a,b]$ and $m-1 < \alpha \leq m, m \in \mathbb{N}$ is defined as:

$${}_a^C D_t^\alpha u(t) = {}_aJ_t^{m-\alpha} D_t^m u(t)$$

Thus for $\alpha = m, m \in \mathbb{N}_0$, and $u \in C^m[a,b]$, we have for all $a \leq t \leq b$

$$ {}_a^C D_t^0 u(t) = u(t) \quad ; \quad {}_a^C D_t^m u(t) = D_t^m u(t) = \frac{d^m u(t)}{dt^m} $$

Note that: [12,13,18,21]

i. For $\alpha \geq 0$ and $\beta > 0$, then ${}_a J_t^\alpha (t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (t-a)^{\beta+\alpha-1}$.

ii. For all $\alpha \geq 0$, $\beta \geq 0$ and $u(t) \in C_\gamma[a,b], \gamma \geq -1$, then: ${}_a J_t^\alpha \, {}_a J_t^\beta u(t) = {}_a J_t^\beta \, {}_a J_t^\alpha u(t) = {}_a J_t^{\alpha+\beta} u(t)$

iii. ${}_a^R D_t^\alpha \mathcal{C} = \mathcal{C} \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}$ and ${}_a^C D_t^\alpha \mathcal{C} = 0$; $\mathcal{C}$ is any constant; ($\alpha \geq 0, \alpha \notin \mathbb{N}$)

iv. ${}_a^R D_t^\alpha u(t) = D_t^m \, {}_a J_t^{m-\alpha} u(t) \neq {}_a J_t^{m-\alpha} D_t^m u(t) = {}_a^C D_t^\alpha u(t)$; $m = \lceil \alpha \rceil$.

v. Assume that $u \in C_{-1}^m[a,b]$; $\alpha \geq 0, \alpha \notin \mathbb{N}$ and $m = \lceil \alpha \rceil$ then ${}_a^C D_t^\alpha u(t)$ is continuous on $[a,b]$, and $[{}_a^C D_t^\alpha u(t)]_{t=a} = 0$.

vi. Let $\alpha \geq 0$, $m = \lceil \alpha \rceil$ and $u \in C^m[a,b]$, then, the relation between the Caputo derivative and R-L integral are formed:

$$ {}_a^C D_t^\alpha \, {}_a J_t^\alpha u(t) = u(t) \quad ; \quad a \leq t \leq b $$

$$ {}_a J_t^\alpha \, {}_a^C D_t^\alpha u(t) = u(t) - \sum_{k=0}^{m-1} \frac{u^{(k)}(a)}{k!} (t-a)^k $$

vii. ${}_a^C D_t^\alpha u(t) = {}_a^R D_t^\alpha \left[u(t) - T_{m-1}[u;a] \right]$, $(m-1 < \alpha \leq m)$ and $T_{m-1}[u;a]$ denotes the Taylor polynomial of degree $m-1$ for the function u, centered at a.

viii. Let $\alpha \geq 0$; $m = \lceil \alpha \rceil$ and for $u(t) = (t-a)^\beta$ for some $\beta \geq 0$. Then:

$$ {}_a^C D_t^\alpha u(t) = \begin{cases} 0 & if\ \beta \in \{0,1,2,\cdots,m-1\} \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} (t-a)^{\beta-\alpha} & if\ \beta \in \mathbb{N}\ and\ \beta \geq m \\ & or\ \beta \notin \mathbb{N}\ and\ \beta > m-1 \end{cases} $$

We adopt Caputo's definition, which is a modification of the R-L definition and has the advantage of dealing properly with initial value problem, for the concept of the fractional derivative, [17].

Solution Technique for IFDE of V-H type:

In this section, a new algorithm for solving multi-higher fractional order nonlinear IDE's of V-H type with variable coefficients applying weighted residual technique with use trial function φ_r , ($r = \overline{0:N}$) has been presented. Here, we approximate the unknown function $u(t)$ in equation (1) with initial conditions (2) by $u_N(t)$:

$$ u(t) \cong u_N(t) = \hat{\varphi}_\mu(t) + \sum_{j=1}^{N} \hat{u}_j \, \varphi_j(t) \tag{3} $$

Where $\hat{\varphi}_\mu(t)$ are satisfies μ initial conditions and $\varphi_j(t)$ are usually chosen as a power function on a domain that satisfy the homogenous conditions. That is, $u_N(t)$ automatically satisfies the initial conditions, in this case a so-called interior method is obtained [9]. And, $\hat{u}_j\,'s$ are undetermined constant coefficients for all $j = 1,2,\cdots,N$. Also, for first basic in equation (3), $\hat{\varphi}_\mu(t)$, we choose the Taylor's expansion for $u(t)$ around lower limit point a of order $\mu - 1$, i.e.

$$ \hat{\varphi}_\mu(t) = T_{\mu-1}[u;a] = \sum_{k=0}^{\mu-1} \frac{u_k}{k!} (t-a)^k $$

It is easy to see that the trial function satisfies the homogenous conditions at lower limit point $t = a$ by using shifted power function $\varphi_j(t) = (t-a)^{j+\mu-1}$ for all $j = 1,2,\cdots,N$ and $\hat{\varphi}_\mu(t)$ satisfies all initial conditions in our problem (1-2), i.e.

$$\left[D_t^k \hat{\varphi}_\mu(t)\right]_{t=a} = u_k \; ; \quad k = 0,1,2,\cdots,\mu-1$$

$$\left[D_t^k \varphi_j(t)\right]_{t=a} = \left[D_t^k (t-a)^{j+\mu-1}\right]_{t=a} = \left[\frac{\Gamma(j+\mu)}{\Gamma(j+\mu-k)}(t-a)^{j+\mu-k-1}\right]_{t=a} = 0$$

for $k = 0,1,2,\cdots,\mu-1$ and $j = 1,2,\cdots,N$. Any nonlinear IFDE of V-H type (1) may be written in the form:

$$\boldsymbol{N}[u(t)] = f(t)\,, \quad t \in [a,\mathcal{T}] \tag{4}$$

Where $\boldsymbol{N}[\cdot]$ denote nonlinear operator, which is defined as:

$$\boldsymbol{N}[u(t)] = \sum_{i=0}^{n} p_i(t)\, {}_a^C D_t^{\alpha_i} u(t) - \lambda \sum_{\ell=0}^{m} \int_a^t \mathcal{K}_\ell(t,s)\, \mathcal{H}_\ell\left(s\,, {}_a^C D_s^{\beta_\ell} u(s)\right) ds\,, \;\; p_n(t) = 1$$

Now, by substituting the approximate solution $u_N(t)$, which is formed in equation (3), into equation (4), we obtain the residual (Error) function

$$E_N\big(t;\widehat{U}\big) = \boldsymbol{N}\left[\hat{\varphi}_\mu(t) + \sum_{j=1}^{N} \hat{u}_j\, \varphi_j(t)\right] - f(t) \tag{5}$$

where $\widehat{U} = [\hat{u}_1, \hat{u}_2, \cdots, \hat{u}_N]$. The function $u(t)$ is an admissible solution to the problem (1-2) if and only if make the residual function $E_N\big(t;\widehat{U}\big)$ as small as possible.

In this paper, we consider a set of N test functions ψ_r on $[a,\mathcal{T}]$. The smallness of E_N is enforced by demanding that

$$\langle\, \psi_r(t)\;, E_N(t;\hat{u}_1,\hat{u}_2,\cdots,\hat{u}_N)\,\rangle_{\,M,W} = 0 \quad , \qquad \forall r \le N \tag{6}$$

Where $\langle\,,\rangle_{M,W}$ is the discrete inner product for the set of M- quadrature points with associated weights W, [17]. The main points here, is how to find the coefficients $\hat{u}_j$ $(j = 1,2,\cdots,N)$ of $u_N(t)$ in equation (3) such that $E_N\big(t;\widehat{U}\big)$ is minimized. This can be depended on choice the test functions ψ_r in (6). The technique described by equation (6) is called discrete weighted residual methods. We now describe a few well known methods of the weighted residual methods to determine the arbitrary parameters $\hat{u}_j\,, (j = 1,2,\cdots,N)$. according to the choices of test functions ψ_r in equation (6).

A. Collocation Method:

In this method the test functions are taking from the family of Dirac Delta-functions in the domain [15]. That is, $\psi_r(t) = \delta(t-t_r)$, for all $r \le N$ and $t \in [a,\mathcal{T}]$. Hence, the residual equation (6) equivalent to :

$$E_N(t_r;\hat{u}_1,\hat{u}_2,\cdots,\hat{u}_N) = 0\,;\; \forall r \le N \tag{7}$$

We choose for our collocation points the zeros of $T_N(t)$, namely Gauss-Chebyshev points:

$$x_r = -\cos\left(\frac{2r-1}{2N}\pi\right) \;\; ; \;\; t_r = \frac{\mathcal{T}-a}{2}x_r + \frac{\mathcal{T}+a}{2} \;\; ; \, r = \overline{1{:}N}$$

Setting the points $t = t_r$ $(r = 1,2,\cdots,N)$ and using the residual equation (5) with equation (7), we obtain N-equations:

$$\boldsymbol{N}\left[\hat{\varphi}_\mu(t_r) + \sum_{j=1}^{N} \hat{u}_j\, \varphi_j(t_r)\right] = f(t_r) \tag{8}$$

Thus, from the definition of nonlinear operator $\boldsymbol{N}$ we have for each $r = 1,2,\cdots,N$ the nonlinear equations:

$$\sum_{i=0}^{n} p_i(t_r) \left\{ {}_a^C D_t^{\alpha_i} \left[\hat{\varphi}_\mu(t) + \sum_{j=1}^{N} \hat{u}_j\, \varphi_j(t) \right] \right\}_{t=t_r} - \lambda \sum_{\ell=0}^{m} \int_a^{t_r} \mathcal{K}_\ell(t_r, s)\, \mathcal{H}_\ell \left(s\,, {}_a^C D_s^{\beta_\ell} \left[\hat{\varphi}_\mu(s) + \sum_{j=1}^{N} \hat{u}_j\, \varphi_j(s) \right] \right) ds = f(t_r) \tag{9}$$

where

$${}_a^C D_t^{\gamma}\, \hat{\varphi}_\mu(t) = \sum_{k=1}^{\mu-1} \frac{u_k}{k!} * \begin{cases} \dfrac{\Gamma(k+1)}{\Gamma(k+1-\gamma)} (t-a)^{k-\gamma} & if\ \lceil\gamma\rceil \in \{1,2,\cdots,k\} \\ 0 & if\ \lceil\gamma\rceil \geq k+1 \end{cases} \tag{10}$$

$${}_a^C D_t^{\gamma} \left[\sum_{j=1}^{N} \hat{u}_j\, \varphi_j(t) \right] = \sum_{j=1}^{N} \hat{u}_j \begin{cases} \dfrac{\Gamma(j+\mu)}{\Gamma(j+\mu-\gamma)} (t-a)^{j+\mu-\gamma-1} & if\ \lceil\gamma\rceil \in \{1,2,\cdots,j+\mu-1\} \\ 0 & if\ \lceil\gamma\rceil \geq j+\mu \end{cases} \tag{11}$$

also

$$\lceil\gamma\rceil = \begin{cases} n_i & if\ \gamma = \alpha_i \quad \text{for } i = 0,1,\cdots,n \\ m_\ell & if\ \gamma = \beta_\ell \quad \text{for } \ell = 0,1,\cdots,m \end{cases}$$

Usually, this system (9) of N-algebraic nonlinear equations will have to be solved by some kind of iterative method. It is therefore a marked disadvantage of the above collocation method that the N definite integrals in (9) need to be evaluated at each step of the iteration. In practice, these integrals will have to be approximated by a numerical integration technique say Gauss-Legender quadrature formula [8]. Thus

$$\sum_{i=0}^{n} p_i(t_r) \left\{ {}_a^C D_t^{\alpha_i} \left[\hat{\varphi}_\mu(t) + \sum_{j=1}^{N} \hat{u}_j\, \varphi_j(t) \right] \right\}_{t=t_r} - \lambda\, \frac{t_r - a}{2} \sum_{\ell=0}^{m} \sum_{q=1}^{Q} \varpi_q\, \mathcal{K}_\ell\big(t_r, s_q^r\big) * \mathcal{H}_\ell \left(s_q^r\,, \left\{ {}_a^C D_s^{\beta_\ell} \left[\hat{\varphi}_\mu(s) + \sum_{j=1}^{N} \hat{u}_j\, \varphi_j(s) \right] \right\}_{s=s_q^r} \right) = f(t_r) \tag{12}$$

where

$$s_q^r = \left(\frac{t_r - a}{2} \right) z_q + \left(\frac{t_r + a}{2} \right) \quad ; \quad \varpi_q = \frac{2}{(1 - z_q^2)[P_Q'(z_q)]^2}$$

and z_q are the q-th zeros of Q-th degree Legender polynomial $P_Q(t)$.

From equation (12) obtain the system of N-equations with N-unknown coefficients $\widehat{U} = [\hat{u}_1, \hat{u}_2, \cdots, \hat{u}_N]$. To find these coefficients, solve the resulting system of N-algebraic nonlinear equation (12) by applying Newton's Method for nonlinear equations [8,10]. Finally, substitute the finding values of coefficients in to equation (3) we obtain the approximate solution of equation (1).

The Algorithm (CM): The approximate solution of IFDE's of V-H type (1) with use of WR Collocation method can be summarized in the following steps:

Step1:

a. input N (Number of approximate coefficients) and Q (Number of degree of Legender polynomial)
b. Set $t_r = \frac{\mathcal{T}-a}{2} x_r + \frac{\mathcal{T}+a}{2}$; x_r's are Gauss –Chebyshev points

$$x_r = -\cos\left(\frac{2r-1}{2N}\pi\right)\ ;\ r = \overline{1:N}.$$

c. Put $s_q^r = \left(\frac{t_r - a}{2}\right) z_q + \left(\frac{t_r + a}{2}\right)$; z_q are the q-th zeros of Q-th degree of Legender polynomial. Also $\varpi_q = \frac{2}{(1-z_q^2)[P_Q'(z_q)]^2}$ where $r = \overline{1:N}$ and $q = \overline{1:Q}$.

Step2: For all $r = 1,2,\cdots,N$:

a. Compute ${}_a^C D_t^{\gamma} \hat{\varphi}_\mu(t)$ and ${}_a^C D_t^{\gamma} \varphi_j(t)$; $j = 1,2,\cdots,N$, at $t = t_r$ and $t = s_q^r$ $(q = \overline{1:Q})$ from equation (10 and 11) respectively, for each $\gamma = \alpha_i (i = \overline{0:n})$ and $\gamma = \beta_\ell$ $(\ell = \overline{0:m})$.

b. Determine $p_i(t_r)$, $(i = \overline{0:n})$ and $f(t_r)$ in given the form p_i and f in equation (1).

c. Evaluate $\mathcal{K}_\ell(t_r, s_q^r)$ for all $\ell = \overline{0:m}$ and $q = \overline{1:Q}$.

d. Calculate $\mathcal{H}_\ell\left(s_q^r\ ,\left[{}_a^C D_s^{\beta_\ell} u_N\right]_{s=s_q^r}\right)$ for all $\ell = \overline{0:m}$ and $q = \overline{1:Q}$ where u_N is defined in equ. (3).

Step3: put the results of steps (1 and 2) in equation (12) to obtain N-algebraic nonlinear equations for N-unknown approximate coefficients $\hat{U}$.

Step4: solve the nonlinear system of step (3) by using Newton's method for finding $\hat{u}_1, \hat{u}_2, \cdots, \hat{u}_N$.

Step5: Finally, substitute $\hat{u}_i (i = \overline{1:N})$ in the equation (3), to obtain the approximate solution of (1).

B. Sub-domain Method:

This method can be considered as a modification of the collocation method. The idea is to force the weighted residual equal to zero not just at fixed points in the domain, but over the various subsections of the domain [5]. Here the domain is divided in N subdomains $D_r\ ; r = 1,2,\cdots,N$. Note that the choice of the subdomains is free. We take the test functions as $\psi_r(t) = 1$ where $t \in D_r = [t_{r-1}, t_r]$ and equal to zero otherwise. Hence, the residual equation (6) after using discrete inner product definition is equivalent to

$$\sum_{d=0}^{M} w_d \psi_r(\xi_d^r) E_N(\xi_d^r; \hat{u}_1, \hat{u}_2, \cdots, \hat{u}_N) = 0\ ;\ r = \overline{1:N} \tag{13}$$

Here, choose the Chebyshev-Gauss-Labotto collocation points with corresponding weights associated with Chebyshev polynomial can be computed [2]:

$$\xi_d^r = \frac{t_r - t_{r-1}}{2} x_d + \frac{t_r + t_{r-1}}{2}\ ;\ x_d = -\cos\left(\frac{d\pi}{M}\right)$$

For all $r = 1,2,\cdots,N$ and $d = 0,1,\cdots,M$ and weighted are

$$w_d = \left\{\frac{\pi}{2M} \quad if\ d = 0\ or\ d = M \quad \text{and} \quad \frac{\pi}{M} \quad \text{otherwise}\right\}$$

Now, we obtain from equations (5) and (13) the following N-algebraic nonlinear equations:

$$\sum_{d=0}^{M} W_d \left\{ \boldsymbol{N}\left[\hat{\varphi}_\mu(\xi_d^r) + \sum_{j=1}^{N} \hat{u}_j \varphi_j(\xi_d^r)\right] - f(\xi_d^r) \right\} = 0 \tag{14}$$

For all $\xi_0^r, \xi_1^r, \cdots, \xi_M^r \in D_r = [t_{r-1}, t_r], (r = 1,2,\cdots,N)$ where

$$\boldsymbol{N}\left[\hat{\varphi}_\mu(\xi_d^r) + \sum_{j=1}^{N} \hat{u}_j \varphi_j(\xi_d^r)\right] = \sum_{i=0}^{n} p_i(\xi_d^r) \left\{ {}_a^C D_t^{\alpha_i}\left[\hat{\varphi}_\mu(t) + \sum_{j=1}^{N} \hat{u}_j\, \varphi_j(t)\right]\right\}_{t=\xi_d^r}$$

$$-\lambda \sum_{\ell=0}^{m} \int_a^{\xi_d^r} \mathcal{K}_\ell(\xi_d^r, s)\, \mathcal{H}_\ell\left(s\ , {}_a^C D_s^{\beta_\ell}\left[\hat{\varphi}_\mu(s) + \sum_{j=1}^{N} \hat{u}_j\, \varphi_j(s)\right]\right) ds$$

For summation term we can use equation (10 and 11) to evaluate it and for integral terms applying Gauss-Legender quadrature formula [8]. Finally, equation (14) becomes:

$$\sum_{d=0}^{M} W_d \left\{ \sum_{i=0}^{n} p_i(\xi_d^r) \left\{ {}_a^C D_t^{\alpha_i} \left[\hat{\varphi}_\mu(t) + \sum_{j=1}^{N} \hat{u}_j \, \varphi_j(t) \right] \right\}_{t=\xi_d^r} \right.$$

$$\left. -\lambda \, \frac{\xi_d^r - a}{2} \sum_{\ell=0}^{m} \sum_{q=1}^{Q} \varpi_q \, \mathcal{K}_\ell(\xi_d^r, s_q^r) \, \mathcal{H}_\ell \left(s_q^r \, , \left\{ {}_a^C D_s^{\beta_\ell} \left[\hat{\varphi}_\mu(s) + \sum_{j=1}^{N} \hat{u}_j \, \varphi_j(s) \right] \right\}_{s=s_q^r} \right) \right\} = \sum_{d=0}^{M} w_d f(\xi_d^r) \qquad (15)$$

where

$$s_q^r = \left(\frac{\xi_d^r - a}{2} \right) z_q + \left(\frac{\xi_d^r + a}{2} \right) \quad ; \quad \varpi_q = \frac{2}{(1 - z_q^2)[P_Q'(z_q)]^2}$$

and z_q are the q-th zeros of Q-th degree Legender polynomial $P_Q(t)$.

From equation (15) obtain the system containing N-algebraic nonlinear equations with N-unknown coefficients $\hat{u}_j(j = \overline{1:N})$. Applying Newton iteration nonlinear technique to obtain these coefficients and substitute the values of $\hat{u}_j$ into equation (3) to obtain the approximate solution of equation (1).

The Algorithm (SDM): The Sub-Domain method in WR technique can be summarized in the following steps to approximate the solution of equation (1):

Step1:

a. Input N(Number of approximate coefficients which is equal to number of subdomains $D_r(r = \overline{1:N})$ and Q(Number of degree of Legendre polynomial) also, M(Number of Chebyshev-Gauss-Labotto points).

b. Let $D_r = [t_{r-1}, t_r]$ as equal length by $h = \frac{\mathcal{T} - a}{N}$; $a = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = \mathcal{T}$; $t_r = t_{r-1} + h$.

c. Set $\xi_d^r = \frac{t_r - t_{r-1}}{2} x_d + \frac{t_r + t_{r-1}}{2}$; x_d's are Chebyshev-Gauss-Labotto points $x_d = -\cos\left(\frac{d\pi}{M}\right)$ for all $d = \overline{0:M}$ and $r = \overline{1:N}$. Also, put $w_d = \frac{\pi}{2M}$ for $d = 0$ or $d = M$ and otherwise $w_d = \frac{\pi}{M}$.

d. Put $s_q^r = \left(\frac{\xi_d^r - a}{2}\right) z_q + \left(\frac{\xi_d^r + a}{2}\right)$; z_q are q-th zeros of Q-th degree of Legendre polynomial. Also ; $\varpi_q = \frac{2}{(1 - z_q^2)\left[P_Q'(z_q)\right]^2}$ where $r = \overline{1:N}$ and $q = \overline{1:Q}$.

Step2: For all $r = 1,2, \cdots, N$:

a. Compute ${}_a^C D_t^{\gamma} \hat{\varphi}_\mu(t)$ and ${}_a^C D_t^{\gamma} \varphi_j(t)$; $j = 1,2, \cdots, N$, at $t = \xi_d^r$ $(d = \overline{0:M})$ and s_q^r $(q = \overline{1:Q})$ from equations (10 and 11) respectively, for each $\gamma = \alpha_i (i = \overline{0:n})$ and $\gamma = \beta_\ell$ $(\ell = \overline{0:m})$.

b. Determine $p_i(\xi_d^r)$, $(i = \overline{0:n})$ and $f(\xi_d^r)$ for all $d = \overline{0:M}$ which is given in equation (1) p_i and f.

c. Evaluate $\mathcal{K}_\ell(\xi_d^r, s_q^r)$ for all $d = \overline{0:M}$ and $q = \overline{1:Q}$ which is gives in equation(1) $\mathcal{K}_\ell$ for all $\ell = 0,1, \cdots, m$.

d. Calculate $\mathcal{H}_\ell \left(s_q^r \, , \left[{}_a^C D_s^{\beta_\ell} u_N \right]_{s = s_q^r} \right)$ for all $\ell = \overline{0:m}$ and $q = \overline{1:Q}$ where $u_N(t)$ is equal to $\hat{\varphi}_\mu(t) + \sum_{j=1}^{N} \hat{u}_j \varphi_j(t)$.

Step3: Substitute the results of step (1 and 2) in equation (15), we get N-algebraic nonlinear equations for N-unknown coefficients $\widehat{U}$.

Step4: Solve the nonlinear system of step (3) by using Newton's method for finding $\hat{u}_1, \hat{u}_2, \cdots, \hat{u}_N$.

Step5: Putting $\hat{u}_i(i = \overline{1:N})$ in the equation (3), to obtain the approximate solution of (1).

C. Least Square Method:

In this method we take the test function $\psi_r(t), r=\overline{1:N}$, as the derivative of the residual function $E_N\big(t,\widehat{U}\big)$, in equation (5), with respect to the unknown coefficient approximations $\hat{u}_r, r=\overline{1:N}$, as follow:

$$\psi_r(t)=\frac{\partial}{\partial\hat{u}_r}E_N(t;\hat{u}_1,\hat{u}_2,\cdots,\hat{u}_N)=\sum_{i=0}^{n}p_i(t)\,{}_a^CD_t^{\alpha_i}\varphi_r(t)$$

$$-\lambda\sum_{\ell=0}^{m}\int_a^t\mathcal{K}_\ell(t,s)\,\frac{\partial}{\partial\hat{u}_r}\mathcal{H}_\ell\left(s\,,{}_a^CD_s^{\beta_\ell}\left[\hat{\varphi}_\mu(s)+\sum_{j=1}^{N}\hat{u}_j\,\varphi_j(s)\right]\right)ds \qquad (16)$$

By applying Gauss-Legendre quadrature formula for integration in (16) at each points $t=t_d$, equation (16) may be restated as:

$$\psi_r(t_d)=\sum_{i=0}^{n}p_i(t_d)\left\{{}_a^CD_t^{\alpha_i}\varphi_r(t)\right\}_{t=t_d}$$

$$-\lambda\,\frac{t_d-a}{2}\sum_{\ell=0}^{m}\sum_{q=1}^{Q}\varpi_q\,\mathcal{K}_\ell\big(t_d,s_q^d\big)\,\frac{\partial}{\partial\hat{u}_r}\mathcal{H}_\ell\left(s_q^d\,,\left\{{}_a^CD_s^{\beta_\ell}\left[\hat{\varphi}_\mu(s)+\sum_{j=1}^{N}\hat{u}_j\,\varphi_j(s)\right]\right\}_{s=s_q^d}\right) \qquad (17)$$

where

$$s_q^d=\left(\frac{t_d-a}{2}\right)z_q+\left(\frac{t_d+a}{2}\right)\quad;\quad\varpi_q=\frac{2}{(1-z_q^2)[P_Q'\big(z_q\big)]^2}$$

And z_q are the q-th zeros of Q-th degree Legendre polynomial $P_Q(t)$.

Now, after substitute equation (17) and (5) into equation (6) and using discrete inner product definition for the set of M-chebyshev –Gauss – Labotto points x_d with corresponding weights associated of Chebyshev polynomial:

$$x_d=-\cos\left(\frac{d\pi}{M}\right)\ ;\ t_d=\frac{\mathcal{T}-a}{2}x_d+\frac{\mathcal{T}+a}{2}$$

$$w_d=\left\{\frac{\pi}{2M}\qquad if\ d=0\ or\ d=M\quad\text{and}\quad\frac{\pi}{M}\qquad otherwise\right\}$$

Thus; we obtain N-algebraic nonlinear equations:

$$\sum_{d=0}^{M}w_d\psi_r(t_d)\left(\boldsymbol{N}\left[\hat{\varphi}_\mu(t_d)+\sum_{j=1}^{N}\hat{u}_j\,\varphi_j(t_d)\right]-f(t_d)\right)=0\ ;\ r\le N \qquad (18)$$

where

$$\boldsymbol{N}[u_N(t_d)]\ =\sum_{i=0}^{n}p_i(t_d)\,\left\{{}_a^CD_t^{\alpha_i}u_N(t)\right\}_{t=t_d}$$

$$-\lambda\,\frac{t_d-a}{2}\sum_{\ell=0}^{m}\sum_{q=1}^{Q}\varpi_q\,\mathcal{K}_\ell\big(t_d,s_q^d\big)\,\mathcal{H}_\ell\left(s_q^d\,,\left[{}_a^CD_s^{\beta_\ell}u_N(s)\right]_{s=s_q^d}\right) \qquad (19)$$

From equation (18), we have the system of N-algebraic nonlinear equations with N-unknown coefficients $\hat{u}_j(j=\overline{1:N})$. Use Newton iteration nonlinear technique to obtain $\hat{u}_j$'s and substitute these values into equation (3) to obtain the approximate solution of equation (1).

The Algorithm of (LSM): The approximate solution of IFDE's of V-H type (1) using Least Square method in WR technique can be summarized in the following stages:

Step1:

a. Input N(Number of approximate coefficients) and Q(Number of degree of Legendre polynomial). Also, M(Number of Chebyshev-Gauss-Labotto points).

b. Set $t_d = \frac{\mathcal{T}-a}{2} x_d + \frac{\mathcal{T}+a}{2}$; x_d's are Chebyshev-Gauss-Labotto points $x_d = -\cos\left(\frac{d\pi}{M}\right)$ for all $d = 0,1,\cdots,M$. Also, let $w_d = \frac{\pi}{2M}$ for $d = 0$ or M and otherwise $w_d = \frac{\pi}{M}$.

c. Put $s_q^d = \left(\frac{t_d-a}{2}\right) z_q + \left(\frac{t_d+a}{2}\right)$; $\varpi_q = \frac{2}{(1-z_q^2)[P_Q'(z_q)]^2}$

Where z_q are q-th zeros of Q-th degree of Legendre polynomial. Where $q = \overline{1:Q}$ and $d = \overline{0:M}$.

Step2: For all $r = 1,2,\cdots,N$

a. Compute ${}_a^C D_t^\gamma \hat{\varphi}_\mu(t)$ and ${}_a^C D_t^\gamma \varphi_j(t)$; $j = 1,2,\cdots,N$, at points $t = t_d$ and s_q^d $(q = \overline{1:Q})$ from equations (10 and 11) respectively, for each $\gamma = \alpha_i (i = \overline{0:n})$ and $\gamma = \beta_\ell$ $(\ell = \overline{0:m})$, for all $d = 0,1,2,\cdots,M$.

b. Determine $p_i(t_d)$, $(i = \overline{0:n})$ and $f(t_d)$ for all $d = \overline{0:M}$ which are given in equation (1) p_i and f.

c. Evaluate $\mathcal{K}_\ell(t_d, s_q^d)$ for all $d = \overline{0:M}$ and $q = \overline{1:Q}$,which are gives in equation(1) $\mathcal{K}_\ell$ for all $\ell = 0,1,\cdots,m$.

d. Calculate $\mathcal{H}_\ell\left(s_q^d , \left[{}_a^C D_s^{\beta_\ell} u_N\right]_{s=s_q^d}\right)$ for all $\ell = \overline{0:m}$ and $q = \overline{1:Q}$ where $u_N(t)$ is equal to $\hat{\varphi}_\mu(t) + \sum_{j=1}^N \hat{u}_j \varphi_j(t)$.

Step3: Use step2 into equation (17) to prepare $\psi_r(t_d)$for all $r = \overline{1:N}$ and $d = \overline{0:M}$.

Step4: Use step 2 into equation (19) to prepare $\boldsymbol{N}[u_N(t_d)]$ for all $d = \overline{0:M}$.

Step5: Substitute the results of steps (2, 3, and 4) in equation (18), we get N-algebraic nonlinear equations for N-unknown coefficients $\widehat{U}$.

Step6: Solve the nonlinear system of step (5) by applying Newton's method for finding $\hat{u}_1, \hat{u}_2, \cdots, \hat{u}_N$.

Step7: Putting $\hat{u}_i (i = \overline{1:N})$ in the equation (3) to obtain the approximate solution of equation (1).

D. Galerkin Method:

This method may be viewed as a modification of the least square method. Rather than using the derivative of the residual function with respect to the unknown $\hat{u}_i (i = \overline{1:N})$, the derivative of the approximating function is used, [7]. That is, we take the test function $\psi_r(t), r = \overline{1:N}$, as the derivative of the approximate function $u_N(t)$, equation (3), with respect to the unknown $\hat{u}_i (i = \overline{1:N})$, as follows

$$\psi_r(t) = \frac{\partial}{\partial \hat{u}_r} u_N(t) = (t-a)^r \ ; \ r \le N \tag{20}$$

Applying same stages as in least square method, we obtain N-algebraic nonlinear equations:

$$\sum_{d=0}^{M} w_d \, (t_d - a)^r \left\{ \boldsymbol{N}\left[\hat{\varphi}_\mu(t_d) + \sum_{j=1}^{N} \hat{u}_j \, \varphi_j(t_d) \right] - f(t_d) \right\} = 0; \ \ r \le N \tag{21}$$

where $\boldsymbol{N}[u_N(t_d)]$ is explain in equation (19), and

$$x_d = -\cos\left(\frac{d\pi}{M}\right) \ ; \ t_d = \frac{\mathcal{T}-a}{2} x_d + \frac{\mathcal{T}+a}{2} \ , d = \overline{0:M}$$

with $w_d = \frac{\pi}{2M}$ where $d = 0$ or M and otherwise $w_d = \frac{\pi}{M}$.

Furthermore for all $q = \overline{1:Q},\ s_q^d = \left(\frac{t_d-a}{2}\right) z_q + \left(\frac{t_d+a}{2}\right);\ \ \varpi_q = \frac{2}{(1-z_q^2)[P_Q'(z_q)]^2}$; and z_q are the q-th zeros of Q-th degree Legendre polynomial $P_Q(t)$. From equation (21) we have the system of N-algebraic nonlinear equations with N-unknown coefficients $\hat{u}_j(j = \overline{1:N})$.Use Newton iteration nonlinear technique to obtain $\hat{u}_j$'s and substitute these values into equation (3) to obtain the approximate solution of equation (1).

The Algorithm (GM): The approximate solution of nonlinear IFDE's of V-H type (1) with use of Galerkin method in WR technique can be summarized by the following:

Step1:

a. Input N(Number of approximate coefficients) and Q(Number of degree of Legendre polynomial). Also, M(Number of Chebyshev-Gauss-Labotto points).

b. Set $t_d = \frac{\mathcal{T}-a}{2} x_d + \frac{\mathcal{T}+a}{2}$; x_d's are Chebyshev-Gauss-Labotto points $x_d = -\cos\left(\frac{d\pi}{M}\right)$ for all $d = 0,1,\cdots,M$. Also, let $w_d = \frac{\pi}{2M}$ for $d = 0$ or M and otherwise $w_d = \frac{\pi}{M}$.

c. Put $s_q^d = \left(\frac{t_d - a}{2}\right) z_q + \left(\frac{t_d + a}{2}\right)$; $\varpi_q = \frac{2}{(1-z_q^2)[P_Q'(z_q)]^2}$

Where $z_q$are q-th zeros of Q-th degree of Legendre polynomial. Where $q = \overline{1:Q}$ and $d = \overline{0:M}$.

Step2: For all $r = 1,2,\cdots,N$

a. Compute ${}_a^C D_t^\gamma \hat{\varphi}_\mu(t)$ and ${}_a^C D_t^\gamma \varphi_j(t)$; $j = 1,2,\cdots,N$, at points $t = t_d$ and s_q^d $(q = \overline{1:Q})$ from equations (10 and 11) respectively, for each $\gamma = \alpha_i (i = \overline{0:n})$ and $\gamma = \beta_\ell$ $(\ell = \overline{0:m})$, for all $d = 0,1,2,\cdots,M$.

b. Determine $p_i(t_d)$, $(i = \overline{0:n})$ and $f(t_d)$ for all $d = \overline{0:M}$ which are given in equation (1) p_i and f.

c. Evaluate $\mathcal{K}_\ell(t_d, s_q^d)$ for all $d = \overline{0:M}$ and $q = \overline{1:Q}$,which are gives in equation(1) $\mathcal{K}_\ell$ for all $\ell = 0,1,\cdots,m$.

d. Calculate $\mathcal{H}_\ell\left(s_q^d, \left[{}_a^C D_s^{\beta_\ell} u_N\right]_{s=s_q^d}\right)$ for all $\ell = \overline{0:m}$ and $q = \overline{1:Q}$ where $u_N(t)$ is equal to $\hat{\varphi}_\mu(t) + \sum_{j=1}^N \hat{u}_j \varphi_j(t)$.

Step3: Use step2 into equation (19) to prepare $\boldsymbol{N}[u_N(t_d)]$ for all $d = \overline{0:M}$.

Step4: Substitute the results of steps (1,2,and 3) in equation (21), we get N-Algebraic nonlinear equations for N-unknown coefficients $\widehat{U}$.

Step5: Solve the nonlinear system of step4 by applying Newton's method for $\hat{u}_1, \hat{u}_2, \cdots, \hat{u}_N$.

Step6: Putting $\hat{u}_i (i = \overline{1:N})$ in the equation (3) to obtain the approximate solution of equation (1)

Numerical Experiment:

In this section, we select some examples in which the exact solution already exists to show the accuracy, efficiency and effectiveness of the proposed algorithms (CM, SM, LSM and GM). All of them were performed on the computer using a program written in MatLab (V. 8). The least square errors in tables are the values of $\sum_{k=0}^R [u(t_k) - u_N(t_k)]^2$, $R \in \mathbb{N}$ at R- selected points t_k and in Newton's method we choose the tolerance $(\varepsilon = 10^{-3})$ and maximum number of iteration is 10 for all examples.

Example 1: We first consider a higher-order nonlinear IFDE of V-H type with variable coefficients:

$$\begin{aligned}{}_0^C D_t^{0.9} u(t) + 3\,{}_0^C D_t^{0.6} u(t) + \sinh(t)\, u(t) \\ = f(t) + \int_0^t \left\{ \frac{s^2 e^t}{16} \left[{}_0^C D_s^{0.75} u(s)\right]^4 + (st^2 - 1)\left[{}_0^C D_s^{0.5} u(s)\right]^2 + e^{s+t} u(s) \right\} ds\end{aligned}$$

$$\begin{aligned}f(t) = \frac{-4}{\Gamma(2.1)} t^{1.1} - \frac{12}{\Gamma(2.4)} t^{1.4} + (1 - 2t^2)\sinh(t) - \frac{2e^t}{\Gamma^4(2.25)}\, t^8 - \frac{16}{\Gamma^2(2.5)}\left\{\frac{t^7}{5} - \frac{t^4}{4}\right\} - 3e^t + 3e^{2t} \\ - 4te^{2t} + 2t^2 e^{2t}\end{aligned}$$

Together with initial condition: $u(0) = 1$; $0 \le t \le 1$. While the exact solution is $u(t) = 1 - 2t^2$.

Now, we try to find the approximate solution $u_N(t)$ by truncated $(N = 2)$, so the approximate solution becomes:

$$u(t) \cong u_2(t) = \hat{\varphi}_\mu(t) + \sum_{j=1}^{2} \hat{u}_j\, t^j$$

Where $\hat{\varphi}_\mu(t) = 1$. Run programs Main CM, Main SDM, Main LSM, and Main GM for $[Q, M] = [20\,, 4]$ to find the unknowns $\hat{u}_j$ ($j = \overline{1:N}$) of the approximate solution of the above problem.

Table-1: presents the values $\hat{u}_j\,'s$ respectively for each techniques.

$\hat{u}_j$ / WRM	$\hat{u}_1$	$\hat{u}_2$
CM	$5.160234e-009$	-2.00000002240853
SDM	$2.165903e-014$	-2.00000000000004
LSM	$-8.973841e-010$	-2.00000000271783
GM	$6.056921e-011$	-2.00000000005105

Thus, the following approximate formulas are obtained:

$$u_2^{CM} = -2.00000002240853t^2 + 5.160234\text{e}-009t + 1.0$$
$$u_2^{SDM} = -2.00000000000004t^2 + 2.165903\text{e}-014t + 1.0$$
$$u_2^{LSM} = -2.00000000271783t^2 - 8.973841\text{e}-010t + 1.0$$
$$u_2^{GM} = -2.00000000005105t^2 + 6.056921\text{e}-011t + 1.0$$

Table-2: presents a comparison between the exact solution $u(t)$ and the approximate solution $u_2(t)$ for all CM, SDM, LSM and GM respectively, depending on the least square error and running time.

t	***Exact solution***	***Discrete Weighted Residual Method***			
		CM	***SDM***	***LSM***	***GM***
0.0	1.0	1.0	1.0	1.0	1.0
0.1	0.98	0.980000000291938	0.98	0.979999999883083	0.980000000005546
0.2	0.92	0.920000000135706	0.92	0.91999999971181	0.920000000010072
0.3	0.82	0.819999999531303	0.82	0.81999999948618	0.820000000013577
0.4	0.68	0.679999998478729	0.68	0.679999999206193	0.68000000001606
0.5	0.5	0.499999996977984	0.5	0.49999999887185	0.500000000017523
0.6	0.28	0.279999995029069	0.280000000000013	0.27999999848315	0.280000000017965
0.7	0.02	0.019999992631837	0.020000000000015	0.019999998040093	0.0200000000173856
0.8	−0.28	−0.280000010213273	−0.279999999999983	−0.28000000245731	−0.279999999984214
0.9	−0.62	−0.620000013506699	−0.619999999999981	−0.62000000300908	−0.619999999986835
1.0	−1.0	−1.0000000172483	−1	−1.00000000361522	−0.999999999990477
L.S.E		$6.7501e-016$	$1.057e-027$	$3.656e-017$	$2.019e-021$
R.Time/Sec		8.317769	16.549583	97.031514	25.215937

Example 2: Consider a higher-order nonlinear IFDE of V-H type with variable coefficients on the closed interval $[0,1]$:

$${}_0^C D_t^{2\alpha} u(t) + 3\,{}_0^C D_t^{\alpha} u(t) + (1+t^2)u(t) = f(t) + \int_0^t \left\{2st^2 e^{\left\{s^{2\beta}\,{}_0^C D_s^{2\beta} u(s)\right\}} + (1+st^2)\left\{{}_0^C D_s^{\beta} u(s)\right\}^2\right\} ds$$

Where

$$f(t) = \frac{2}{\Gamma(3-2\alpha)} t^{2(1-\alpha)} + \frac{6}{\Gamma(3-\alpha)} t^{2-\alpha} + t^4 - 1 - \frac{\Gamma(3-2\beta)}{2} t^2 \left\{ e^{\frac{2}{\Gamma(3-2\beta)} t^2} - 1 \right\}$$
$$- \frac{4}{\Gamma^2(3-\beta)} \left\{ \frac{t^{5-2\beta}}{5-2\beta} + \frac{t^{8-2\beta}}{6-2\beta} \right\}$$

With initial condition: If $0 < \alpha \; and \; \beta < 0.5$ then $u(0) = -1$.

If $0.5 < \alpha \; or \; \beta < 1$ then $u(0) = -1 \;,\; u'(0) = 0$.

While the exact solution is $u(t) = t^2 - 1$

Now, if we take $\alpha = 0.4 \,, \beta = 0.3$ then try to find the approximate solution $u_N(t)$ by truncated $(N = 3)$:

$$u(t) \cong u_3(t) = \hat{\varphi}_\mu(t) + \sum_{j=1}^{3} \hat{u}_j \, t^j$$

Where $\hat{\varphi}_\mu(t) = -1$. Run programs Main CM, Main SDM, Main LSM, and Main GM for $[Q, M] = [20\,, 4]$ to find the unknowns $\hat{u}_j$ $(j = \overline{1{:}N})$ of the approximate solution of the above problem.

Table-3: presents the values $\hat{u}_j\,'s$ respectively for each techniques.

WRM \ $\hat{u}_j$	$\hat{u}_1$	$\hat{u}_2$	$\hat{u}_3$
CM	$-2.970940e-014$	1.00000000000036	$-5.980054e-013$
SDM	$-9.350368e-014$	1.00000000000052	$-6.847944e-013$
LSM	$5.135483e-014$	1.00000000000033	$-1.144734e-012$
GM	$-3.334107e-013$	1.00000000000133	$-1.290256e-012$

Thus, the following approximate formulas are obtained:

$$u_3^{CM} = -5.980054\mathrm{e}-013t^3 + 1.00000000000036t^2 - 2.97094\mathrm{e}-014t - 1$$

$$u_3^{SDM} = -6.847944\mathrm{e}-013t^3 + 1.00000000000052t^2 - 9.350368\mathrm{e}-014t - 1$$

$$u_3^{LSM} = -1.144734\mathrm{e}-012t^3 + 1.00000000000033t^2 + 5.135483\mathrm{e}-014 - 1$$

$$u_3^{GM} = -1.290256\mathrm{e}-012t^3 + 1.00000000000133t^2 - 3.334107\mathrm{e}-013t - 1$$

Table-4: presents a comparison between the exact solution $u(t)$ and approximate solution $u_3(t)$ for all CM, SDM, LSM and GM respectively depending on the least square error and running time.

t	Exact solution	Discrete Weighted Residual Method			
		CM	**SDM**	**LSM**	**GM**
0.0	-1.0	-1.0	-1.0	-1.0	-1.0
0.1	-0.99	-0.99	-0.99	-0.99	-0.99
0.2	-0.96	-0.96	-0.96	-0.96	-0.960000000000024
0.3	-0.91	-0.91	-0.91	-0.91	-0.91
0.4	-0.84	-0.84	-0.84	-0.84	-0.84
0.5	-0.75	-0.75	-0.75	-0.750000000000034	-0.75
0.6	-0.64	-0.640000000000017	-0.640000000000016	-0.640000000000096	-0.64
0.7	-0.51	-0.510000000000049	-0.510000000000045	-0.510000000000193	-0.510000000000023
0.8	-0.36	-0.360000000000099	-0.360000000000092	-0.360000000000331	-0.360000000000075
0.9	-0.19	-0.19000000000017	-0.190000000000161	-0.190000000000518	-0.190000000000162
1.0	0.0	-0.00000000000026	-0.000000000000257	-0.000000000000759	-0.000000000000292
$\boldsymbol{L.S.E}$		$1.1322e-025$	$1.0282e-025$	$1.0014e-024$	$1.184e-025$
$\boldsymbol{R.Time/Sec}$		13.443811	44.453519	311.26121	39.048643

Example (3)*:* Let us consider the nonlinear IFDE of V-H type with variable coefficients, on the closed interval [0,1]:

$${}_0^C D_t^{0.9} u(t) + t\,{}_0^C D_t^{0.7} u(t) + e^t\,{}_0^C D_t^{0.3} u(t) = f(t)$$

$$+ \int_0^t \left\{ (s - e^t) \cos\left(\frac{1 + 3s^2 - u(s)}{2} \right) + (t^2 - s^2)\left\{ {}_0^C D_s^{1.5} u(s) \right\}^2 + \frac{1 - 3ts}{36} \left\{ {}_0^C D_s^{1.2} u(s) \right\}^3 \right\} ds$$

Where

$$f(t)=\frac{6}{\Gamma(2.1)}t^{1.1}-\frac{2}{\Gamma(1.1)}t^{0.1}+\frac{6}{\Gamma(2.3)}t^{2.3}-\frac{2}{\Gamma(1.3)}t^{1.3}-\frac{9}{\Gamma^2(1.5)}t^4+e^t\left(\frac{6}{\Gamma(2.7)}t^{1.7}-\frac{2}{\Gamma(1.7)}t^{0.7}\right)$$
$$+(e^t-t)\sin(t)-\cos(t)-\frac{6}{\Gamma^3(1.8)}\left(\frac{t^{3.4}}{3.4}-\frac{3t^{5.4}}{4.4}\right)+1$$

with initial condition $u(0)=1$, $u'(0)=-2$ and the exact solution is given $u(t)=3t^2-2t+1$.

Now, take $N=2$ and use equation (3). We obtain the approximate solution:

$$u(t)\cong u_2(t)=\hat{\varphi}_\mu(t)+\sum_{j=1}^{2}\hat{u}_j\,t^{j+1}$$

where $\hat{\varphi}_\mu(t)=1-2t$

Table-5: presents the values $\hat{u}_j\,'s$ obtained by running the programs Main CM, Main SDM, Main LSM, and Main GM for $[Q,M]=[20,4]$.

$\hat{u}_j$ / WRM	$\hat{u}_1$	$\hat{u}_2$
CM	2.99999999995559	$2.52175895975e-010$
SDM	2.99999999988745	$3.179353171149e-010$
LSM	2.99999992866921	$3.516081712329e-008$
GM	2.99999999983907	$3.218393466227e-010$

Thus we get the following approximate formula:

$$u_2^{CM}=2.52175895e-010t^3+2.99999999995559t^2-2.0t+1.0$$

$$u_2^{SDM}=3.17935317e-010t^3+2.99999999988745t^2-2.0t+1.0$$

$$u_2^{LSM}=3.516081712e-008t^3+2.99999992866921t^2-2.0t+1.0$$

$$u_2^{GM}=3.2183934662e-010t^3+2.99999999983907t^2-2.0t+1.0$$

Table-6: presents a comparison between the exact solution $u(t)$ and the approximate solution $u_2(t)$ for all CM, SDM, LSM and GM respectively, depending on the least square error and running time.

t	***Exact Solution***	***Discrete Weighted Residual Method***			
		CM	***SDM***	***LSM***	***GM***
0.0	1.0	1.0	1.0	1.0	1.0
0.1	0.83	0.829999999999808	0.829999999999192	0.829999999321853	0.829999999998712
0.2	0.72	0.720000000000241	0.719999999998042	0.719999997428055	0.719999999996137
0.3	0.67	0.670000000002812	0.669999999998455	0.669999994529571	0.669999999994206
0.4	0.68	0.680000000009034	0.680000000002341	0.679999990837366	0.679999999994849
0.5	0.75	0.750000000020421	0.750000000011605	0.749999986562405	0.750000000000000
0.6	0.88	0.880000000038484	0.880000000028157	0.879999981915652	0.880000000011581
0.7	1.07	1.07000000006474	1.0700000000539	1.06999997710807	1.07000000003153
0.8	1.32	1.32000000010069	1.32000000009075	1.31999997235063	1.32000000006178
0.9	1.63	1.63000000014787	1.63000000014061	1.6299999678543	1.63000000010426
1.0	2.0	2.00000000020777	2.00000000020539	1.99999996383003	2.00000000016091
L.S.E		$8.135159e-020$	$7.40382e-020$	4.258707e-015	$4.178437e-020$
R.Time/Sec		10.163356	45.103425	169.06141	28.511534

Example (4): Consider a higher-order nonlinear IFDE of V-H type with variable coefficients on $0\le t\le 1$:

$${}_0^C D_t^{1.8}u(t)+(t+2)u(t)=f(t)+\int_0^t\left\{\left(\frac{t^2+s^2}{6}\right){}_0^C D_s^{1.7}u(s)+\left(\frac{t^2-s^2}{36}\right)\left\{{}_0^C D_s^{1.4}u(s)\right\}^3\right\}ds$$

Where $f(t)=\frac{6}{\Gamma(2.2)}t^{1.2}-\frac{660}{989\Gamma(2.3)}t^{4.3}-\frac{300}{1131\Gamma^3(2.6)}t^{7.8}+t^4+2t^3-2t^2-4t$

With initial condition $u(0)=0$, $u'(0)=-2$, and the exact solution is given $u(t)=t^3-2t$.

Take $N = 3$, and using equation (3), we obtain the approximate solution:

$$u(t) \cong u_3(t) = \hat{\varphi}_\mu(t) + \sum_{j=1}^{3} \hat{u}_j \, t^{j+1}$$

Where

$$\hat{\varphi}_\mu(t) = -2t$$

Run programs Main CM, Main SDM, Main LSM, and Main GM for $[Q, M] = [20\,, 4]$ to find the unknowns $\hat{u}_j$ $(j = \overline{1{:}N})$ of the approximate solution of the above problem.

Table-7: presents the values $\hat{u}_j{}'s$ respectively for each methods.

$\hat{u}_j$ / WRM	$\hat{u}_1$	$\hat{u}_2$	$\hat{u}_3$
CM	$-9.7829835e-010$	1.00000000683578	$-7.98099858e-009$
SDM	$-2.56817724-009$	1.00000000897547	$-8.64345378e-009$
LSM	$-3.3563422e-010$	1.00000000132782	$-5.09375187e-009$
GM	$-5.1589590e-009$	1.0000000142405	$-1.12920716e-008$

Thus, the following approximate formulas are obtained:

$$u_3^{CM} = -7.98099858e-009t^4 + 1.00000000683578t^3 - 9.7829835e-010t^2 - 2.0t$$

$$u_3^{SDM} = -8.643453787e-009t^4 + 1.00000000897547t^3 - 2.56817724-009t^2 - 2.0t$$

$$u_3^{LSM} = -5.093751878e-009t^4 + 1.00000000132782t^3 - 3.3563422e-010t^2 - 2.0t$$

$$u_3^{GM} = -1.12920716e-008t^4 + 1.0000000142405t^3 - 5.1589590e-009t^2 - 2.0t$$

Table-8: presents a comparison between the exact solution $u(t)$ and the approximate solution $u_3(t)$ for all CM, SDM, LSM and GM respectively, depending on the least square error and running time.

t	Exact solution	Discrete Weighted Residual Method			
		CM	**SDM**	**LSM**	**GM**
0.0	0.0	0.0	0.0	0.0	0.0
0.1	-0.199	-0.199000000003745	-0.199000000017571	-0.199000000002538	-0.199000000038478
0.2	-0.392	-0.391999999997215	-0.392000000044753	-0.392000000010953	-0.392000000110502
0.3	-0.573	-0.572999999968127	-0.57300000005881	-0.573000000035615	-0.573000000171279
0.4	-0.736	-0.735999999923351	-0.736000000057751	-0.736000000099121	-0.736000000203119
0.5	-0.875	-0.874999999888914	-0.875000000060327	-0.875000000236291	-0.875000000215432
0.6	-0.984	-0.983999999909996	-0.984000000106035	-0.98400000049417	-0.98400000024473
0.7	-1.057	-1.05700000005093	-1.05700000025511	-1.05700000093203	-1.05700000035463
0.8	-1.088	-1.08800000039521	-1.08800000058855	-1.08800000162136	-1.08800000063583
0.9	-1.071	-1.07100000104547	-1.07100000120808	-1.0710000026459	-1.07100000120616
1.0	-1.0	-1.00000000212351	-1.00000000223616	-1.00000000410157	-1.00000000221053
L.S.E		$5.7884e-018$	$6.8953e-018$	$2.7632e-017$	$7.061e-018$
R.Time/Sec		6.840518	39.455224	89.15063	21.40499

Example 5: Consider a higher-order nonlinear IFDE of V-H type with variable coefficients:

$$ {}_0^C D_t^{\alpha} u(t) + 2(\cos(t) - \sin(t))\, {}_0^C D_t^{\alpha-0.2} u(t) + e^t u(t)$$

$$= f(t) + \int_0^t \left\{ \Gamma^3(3-\beta)(s^2 - t)\left[{}_0^C D_s^{\beta} u(s) \right]^3 + \frac{se^t}{4}\left[{}_0^C D_s^{\beta-0.2} u(s) \right]^2 + \sin\left(\frac{1}{3}s^3 - s + t\right) u(s) \right\} ds$$

where

$$f(t)=\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha}+\frac{4}{\Gamma(3.2-\alpha)}(\cos(t)-\sin(t))t^{2.2-\alpha}+e^{t}(t^{2}-1)-8\left\{\frac{t^{-3\beta+9}}{-3\beta+9}-\frac{t^{-3\beta+8}}{-3\beta+7}\right\}$$
$$-\frac{t^{6.4-2\beta}e^{t}}{(6.4-2\beta)\Gamma^{2}(3.2-\beta)}+\cos\left(\frac{t^{3}}{3}\right)-\cos(t)$$

while the exact solution is $u(t)=t^2-1$ .Together with initial condition:

If $0.2<\alpha$ and $\beta<1$ then $u(0)=-1$

If $1.2<\alpha$ or $\beta<2$ then $u(0)=-1$, $u'(0)=0$

Take $N=2$, and using equation (3). Now, if we take $\boldsymbol{\alpha}=\mathbf{0.65}$, $\boldsymbol{\beta}=\mathbf{0.45}$ then try to find the approximate solution $u_N(t)$ by truncated ($N=2$):

$$u(t)\cong u_2(t)=\hat{\varphi}_\mu(t)+\sum_{j=1}^{2}\hat{u}_j\,t^j$$

Where $\hat{\varphi}_\mu(t)=-1$

Run programs Main CM, Main SDM, Main LSM, and Main GM for $[Q,M]=[20,4]$ to find the unknowns $\hat{u}_j$ ($j=\overline{1:N}$) of the approximate solution of above problem.

Table (9) presents the values $\hat{u}_j$ ‘s respectively, for each method:

WRM \ $\hat{u}_j$	$\hat{u}_1$	$\hat{u}_2$
CM	$-2.32924185869537e-012$	1.00000000001193
SDM	$-6.39956438570086e-012$	1.00000000001517
LSM	$6.16402511877719e-010$	0.999999999996526
GM	$-2.00009942418963e-011$	1.00000000003469

Thus, the following approximate formulas are obtained:

$$u_2^{CM}=1.00000000001193t^2-2.32924185869537\text{e}-012\,t-1$$

$$u_2^{SDM}=1.00000000001517t^2-6.39956438570086\text{e}-012t-1$$

$$u_2^{LSM}=0.999999999996526t^2+6.16402511877719\text{e}-010\,t-1$$

$$u_2^{GM}=1.00000000003469t^2-2.00009942418963e-011t-1$$

Table-10: presents a comparison between the exact solution $u(t)$ and the approximate solution $u_2(t)$ for all CM, SDM, LSM and GM respectively, depending on the least square error and running time.

t	***Exact solution***	***Discrete Weighted Residual Method***			
		CM	***SDM***	***LSM***	***GM***
0.0	−1.0	−1.0	−1.0	−1.0	−1.0
0.1	−0.99	−0.990000000000114	−0.990000000000488	−0.989999999938394	−0.990000000001653
0.2	−0.96	−0.96	−0.960000000000673	−0.959999999876858	−0.960000000002613
0.3	−0.91	−0.909999999999625	−0.910000000000554	−0.909999999815392	−0.910000000002878
0.4	−0.84	−0.839999999999023	−0.840000000000133	−0.839999999753995	−0.84000000000245
0.5	−0.75	−0.749999999998183	−0.749999999999407	−0.749999999692667	−0.750000000001328
0.6	−0.64	−0.639999999997104	−0.639999999998378	−0.639999999631409	−0.639999999999512
0.7	−0.51	−0.509999999995786	−0.509999999997046	−0.50999999957022	−0.509999999997002
0.8	−0.36	−0.35999999999423	−0.359999999995411	−0.359999999509101	−0.359999999993798
0.9	−0.19	−0.189999999992435	−0.189999999993472	−0.189999999448051	−0.189999999989901
1.0	0.0	0.00000000000959	0.00000000000877	0.00000000061292	0.00000000001469
L.S.E		$2.13182e-022$	$1.533322e-022$	$1.44989e-018$	$3.91097e-022$
R.Time/Sec		6.143064	18.349610	8.817975	16.180301

Now, if we take $\boldsymbol{\alpha}=\mathbf{1.5}$, $\boldsymbol{\beta}=\mathbf{0.75}$ then try to find the approximate solution $u_N(t)$ by truncated ($N=3$):

$$u(t) \cong u_3(t) = \hat{\varphi}_\mu(t) + \sum_{j=1}^{3} \hat{u}_j\, t^{j+1}$$

Where $\hat{\varphi}_\mu(t) = -1$. Run programs Main CM, Main SDM, Main LSM, and Main GM for $[Q, M] = [20\,, 4]$ to find the unknowns $\hat{u}_j$ $(j = \overline{1:N})$ of the approximate solution of the above problem.

Table-11: presents the values $\hat{u}_j$ ‘s respectively, for each techniques.

WRM \ $\hat{u}_j$	$\hat{u}_1$	$\hat{u}_2$	$\hat{u}_3$
CM	1.0000000000464	$-4.203090445e-010$	$4.937246193207e-010$
SDM	1.00000000002565	$-9.604280277e-011$	$8.497111030742e-011$
LSM	1.00000000007378	$1.630173683e-009$	$-1.11390685174e-009$
GM	1.00000000010034	$-2.812918908e-010$	$1.992136464470e-010$

Thus, the following approximate formulas are obtained:

$$u_3^{CM} = 4.937246193207e-010t^4 - 4.203090445e-010t^3 + 1.0000000000464t^2 - 1$$

$$u_3^{SDM} = 8.497111030742e-011t^4 - 9.604280277e-011t^3 + 1.00000000002565t^2 - 1$$

$$u_3^{LSM} = -1.11390685174e-009t^4 + 1.630173683e-009t^3 + 1.00000000007378t^2 - 1$$

$$u_3^{GM} = 1.99213646447094e-010t^4 - 2.812918908e-010t^3 + 1.00000000010034t^2 - 1$$

Table-12: presents a comparison between the exact solution $u(t)$ and the approximate solution $u_3(t)$ for all CM, SDM, LSM and GM respectively, depending on the least square error and running time.

t	*Exact solution*	*Discrete Weighted Residual Method*			
		CM	***SDM***	***LSM***	***GM***
0.0	−1.0	−1.0	−1.0	−1.0	−1.0
0.1	−0.99	−0.989999999999907	−0.989999999999831	−0.989999999997743	−0.989999999999258
0.2	−0.96	−0.960000000000716	−0.959999999999606	−0.95999999998579	−0.959999999997918
0.3	−0.91	−0.910000000003173	−0.909999999999596	−0.909999999958367	−0.909999999996951
0.4	−0.84	−0.840000000006836	−0.839999999999868	−0.83999999991238	−0.839999999996849
0.5	−0.75	−0.75000000001008	−0.750000000000282	−0.749999999847402	−0.749999999997627
0.6	−0.64	−0.640000000010095	−0.640000000000499	−0.639999999765683	−0.63999999999882
0.7	−0.51	−0.510000000002886	−0.509999999999973	−0.509999999672146	−0.509999999999487
0.8	−0.36	−0.359999999983271	−0.359999999997954	−0.359999999574386	−0.359999999998208
0.9	−0.19	−0.189999999944887	−0.18999999999349	−0.189999999482674	−0.189999999993086
1.0	0.0	0.000000000119817	0.00000000001457	0.000000000590049	0.000000000018257
L.S.E		$1.794264e-020$	$2.597517e-022$	$9.922289e-019$	$4.157573e-022$
R.Time/Sec		12.767604	48.505053	26.571402	29.168765

Conclusion:

In general, finding the exact solutions of multi-higher order nonlinear integro-fractional differential equations is difficult and needs more mathematical computation or mostly impossible. In this work, the discrete weighted residual method (Collocation, Sub-domain, Least square and Galerkin) is generalized and applied to Volterra-Hammerstein type, which is a good model for nonlinear IFDE’s with non-constant coefficients. It is shown that this procedure transforms such a problem in to a nonlinear algebraic system, which depends on the approximate coefficients.

A Considerable advantage of the method is that the solution is expressed as a truncated power series, and then $u(t)$ can be easily evaluated for arbitrary values of t. To obtain the best approximation, we require more terms from the approximate polynomial expression of functions, i.e. the truncation limit N must be chosen sufficiently large. Also, the truncation limit M (number of terms in discrete WR) must be chosen to be large enough. For each type of discrete WR methods, a computer programs MatLab (V 8) was written and several examples are included for illustration. For the comparison of computing accuracy and speed the least square error and running time of the associative program are given in tabular form.

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